Orientations and Connective Structures on 2-vector Bundles

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ABSTRACT. In [1] a half magnetic monopole is discovered and describes an obstruction to creating a determinant $K(ku) \to ku^*$. In fact it is an obstruction to creating a determinant gerbe map from $K(ku)$ to $K(\mathbb{Z}, 3)$. We describe this obstruction precisely using monoidal categories and define the notion of oriented 2-vector bundles, which removes this obstruction so that we can define a determinant gerbe. We also generalize Brylinskis notion of a connective structure from [4] to 2-vector bundles, in a way compatible with the determinant gerbe.

1 Introduction

In [3] the notion of a charted 2-vector bundle is defined. This is done such that there is a canonical inclusion of charted gerbes (essentially a subset of charted 2-vector bundles of rank 1) into these. They also describe a classifying space $|B\mathrm{Gl}_n(\mathcal{V})|$ of equivalence classes of rank n 2-vector bundles, and this is generalized in [2]. The classifying space of gerbes is $K(\mathbb{Z}, 3)$ (see [4]), and the inclusion of gerbes into 2-vector bundles defines a map of classifying spaces

$$K(\mathbb{Z}, 3) \to |B\mathrm{Gl}_n(\mathcal{V})|. \tag{1}$$

In [1] it is proven that π_3 of this map sends the canonical generator to an element divisible by two (modulo torsion) if n is large enough. An element which multiplied with 2 mod torsion is the image of the generator is what they call a half magnetic monopole. Indeed, this makes sense since a gerbe on S^3 representing the canonical generator of $\pi_3(K(\mathbb{Z}, 3))$ is a mathematical model for a magnetic monopole. As in [1] the existence of the half magnetic monopole provides an obstruction to creating a determinant map

$$|B\mathrm{Gl}_n(\mathcal{V})| \to B(ku^*) \supset |B\mathrm{Gl}_1(\mathcal{V})|,$$

which is the identity on $|B\mathrm{Gl}_1(\mathcal{V})|$ included into $|B\mathrm{Gl}_n(\mathcal{V})|$ in the same way gerbes are included (block sum with and $n-1$ times $n-1$ identity matrix). Here ku^* denotes the invertible components of ku with respect to $\otimes$, i.e. $\{-1, 1\} \times BU$. Indeed, such a map composed with the canonical map

$$B(ku^*) \to K(\mathbb{Z}, 3)$$

would yield a retraction of (1), which is impossible because the half magnetic monopole in π_3 should then be sent to an element which multiplied by 2 is a generator.

[1]The author was funded by the Topology in Norway Project, and would like to thank John Rognes and Bjørn Jahren for many conversations on the subject.

In section one we describe this obstruction in the framework of monoidal categories, and define a natural notion of an orientation on a 2-vector bundle. We also describe a monoidal category $\mathrm{OGl}_n(\mathcal{V})$ such that $|B\mathrm{OGl}_n(\mathcal{V})|$ classifies oriented 2-vector bundles, and we have a forgetful strict monoidal functor from $\mathrm{OGl}_n(\mathcal{V})$ to $\mathrm{Gl}_n(\mathcal{V})$ inducing the map of classifying spaces

$$|B\mathrm{OGl}_n(\mathcal{V})| \to |B\mathrm{Gl}_n(\mathcal{V})|. \tag{2}$$

We then describe the precise obstruction to lifting any map $f\colon X \to |B\mathrm{Gl}_n(\mathcal{V})|$ to the oriented "cover", as a characteristic class in $H^3(X, \mathbb{Z}/2\mathbb{Z})$. Proving that there is a fibration

$$|B\mathrm{OGl}_n(\mathcal{V})| \to |B\mathrm{Gl}_n(\mathcal{V})| \to K(\mathbb{Z}/2\mathbb{Z}, 3).$$

We then describe a canonical lift of the inclusion of gerbes and construct a determinant gerbe functor such that we end up with a retraction

$$K(\mathbb{Z}, 3) \to |B\mathrm{OGl}_n(\mathcal{V})| \to K(\mathbb{Z}, 3).$$

In [4] Brylinski defines a connective structures on gerbes. In section 2 we extend this definition to charted 2-vector bundles, and prove existence and contractibility of choice. This is done such that the functors inducing the maps in equation 2 takes connective structures to connective structure on charted bundles.

2 Orientations and Construction of Determinant Gerbe

Many of the definitions in the following are taken directly from section 2 and section 3 in [3]. However, some are taken from [2], but in the language of monoidal categories - corresponding to bi-categories with one object.

Definition 2.1 Let Σ be the category with

- one object $\mathbf{n} = \{1, \ldots, n\}$ for all non-negative integers $n \in \mathbb{N}_0$,
- and morphisms the permutations Σ_n of $\mathbf{n}$.

Sum $\oplus$ in Σ is defined by disjoint union. More precisely: on objects it is standard addition in $\mathbb{N}_0$ and induced morphisms on $\mathbf{n}+\mathbf{m}$ is defined by order preservingly identifying the first n elements with $\mathbf{n}$ and the last m elements with $\mathbf{m}$.

Product $\otimes$ in Σ is defined by product of sets. More precisely: on objects it is standard multiplication in $\mathbb{N}_0$ and induced morphisms on $\mathbf{nm}$ is defined by identifying the elements in $\mathbf{nm}$ with the elements in $\mathbf{n}\times\mathbf{m}$ using lexicographical ordering. I.e. the first m elements in $\mathbf{nm}$ is identified with $\{1\} \times \mathbf{m}$ the next m with $\{2\} \times \mathbf{m}$ etc.

These operations are strictly associative and has strict units. They are also strictly commutative on the level of objects, but not on the induced morphisms. However, choosing the obvious permutations as coherency isomorphisms it is well-known that we get the structure of a bipermutative category (see e.g. [7]). ♦

Definition 2.2 Let $\mathcal{V}$ be the category with

- one object $\mathbb{C}^n$ for all non-negative integers $n \in \mathbb{N}_0$,
- and morphisms the linear automorphisms $\mathrm{Gl}_n(\mathbb{C})$ of $\mathbb{C}^n$. ♦

The direct sum functor

$$\oplus\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

is defined by $\mathbb{C}^n \oplus \mathbb{C}^m = \mathbb{C}^{n+m}$ on objects and on morphisms by identifying the vector spaces in the standard way. The tensor product functor is defined on objects by $\mathbb{C}^n \otimes \mathbb{C}^m = \mathbb{C}^{nm}$ and on morphisms by using the lexicographical ordering. That is - we identify

$$e_1 \otimes e'_1, \cdots, e_1 \otimes e'_m, e_2 \otimes e'_1, \cdots, e_n \otimes e'_m$$

with the standard basis in $\mathbb{C}^{nm}$, where $e_1, \cdots, e_n$ and $e'_1, \cdots, e'_m$ are the standard bases for $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. As above both operations are strictly associative with units. Since the choices involved in identifying the bases are the same as the choices made for the elements in Σ, the same permutations viewed as matrices may serve as coherency isomorphisms. So again we have a bipermutative category, but also a canonical bipermutative functor

$$S\colon \Sigma \to \mathcal{V}.$$

Definition 2.3 Let $\mathcal{L}$ be the category with

- one object $\mathbb{C}_n$ for all integers $n \in \mathbb{Z}$,
- and morphisms the linear automorphisms $\mathbb{C}_n^* = \mathbb{C}^*$.

We identify the total space of morphisms with $\mathbb{Z} \times \mathbb{C}^*$, and the direct sum functor is then defined by

$$(n, a) \oplus (m, b) = (n + m, ab)$$

and the tensor functor is defined by

$$(n, a) \otimes (m, b) = (nm, a^m b^n).$$

Both products are strictly associative and commutative. So the coherency isomorphisms could be chosen to be identities $(n, 1)$. However, for our purpose it turns out that we need some of the coherency isomorphisms to be different from the identities. More precisely: the coherency twist for the sum

$$\underline{c}\colon \mathbb{C}_n \oplus \mathbb{C}_m \to \mathbb{C}_m \oplus \mathbb{C}_n$$

is defined to be $(n + m, (-1)^{nm})$ and the twist for the product

$$\underline{c}\colon \mathbb{C}_n \otimes \mathbb{C}_m \to \mathbb{C}_m \otimes \mathbb{C}_n$$

is defined to be $(nm, (-1)^{\frac{n(n-1)m(m-1)}{4}})$. As the following lemma shows these choices makes $\mathcal{L}$ into a bipermutative category.

It is convenient to introduce the following terminology: a law or rule in a monoidal category which regardless of coherency isomorphisms holds strictly are called **weakly strict**. This means that the term strict is, as usual, reserved for the laws which have the identity as coherency isomorphism, and we see that strict implies weakly strict. ♦

It was noted by John Rognes and it is a curios fact that there are only two possible E_∞-ring structures on the topological space $\mathbb{Z} \times BU(1) \simeq \mathbb{Z} \times K(\mathbb{Z}, 2)$, and that these arise as the geometric realization of the category above; but with the two different choices of coherency isomorphisms: the trivial making all laws strict and the one we defined.

Lemma 2.4 The above choice of coherency isomorphisms on the category $\mathcal{L}$ makes it bipermutative. ♦

Proof: To check that we indeed have a permutative structure on $\mathcal{L}$ one could tediously check all the diagrams in the definition of a bipermutative category, but a shorter argument using that we know Σ to be bipermutative goes as follows.

Let $\mathcal{L}_+$ be the full sub-category of $\mathcal{L}$ defined by the non-negatively indexed objects. There is a canonical functor $\mathrm{sgn}\colon \Sigma \to \mathcal{L}_+$ which is the obvious bijection on objects and which takes the sign on morphisms. This preserves sum and tensor, and sends coherency isomorphisms to the signs defined in $\mathcal{L}$ as coherency. Because it is a bijection on objects and the fact that Σ is bipermutative makes $\mathcal{L}_+$ bipermutative. The coherency sign in $\mathcal{L}$ for any coherency isomorphism only depends on the objects indices modulo 4. So extending to negatively indexed objects by the same formulas will still satisfy the necessary equations to be bipermutative. □

Construction 2.5 Define the functor

$$\Lambda\colon \mathcal{V} \to \mathcal{L}$$

by $\mathbb{C}^n \mapsto \mathbb{C}_n$ on objects and by taking determinants of morphisms. This preserves both sum and product because the determinant satisfies

$$\det(f \oplus g) = \det(f)\det(g)$$

and

$$\det(f \otimes g) = \det(f)^{\dim(g)} \det(g)^{\dim(f)},$$

where $\dim(f)$ is the dimension of the underlying vector space. The latter can be proved using $(f \otimes g) = (f \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes g)$. This explains the choice of sum and product in $\mathcal{L}$ and we may think of $\mathcal{L}$ as the top exterior power of $\mathcal{V}$ extended to negative dimensions. For this to be a strict bipermutative functor (or even lax bimonoidal) we need that it takes coherency isomorphisms to coherency isomorphisms, and this was why we needed the non-trivial signs as coherency isomorphisms in definition 2.3. So we have a commutative diagram

$$\begin{tikzcd} \Sigma \arrow[rr, "S"] \arrow[dr, "\mathrm{sgn}"'] & & \mathcal{V} \arrow[dl, "\Lambda"] \\ & \mathcal{L} & \end{tikzcd}$$

of bipermutative functors. ♦

We wont use the following explicitly, but it describes very well why this choice of coherency in $\mathcal{L}$ is important.

Lemma 2.6 The induced map on classifying spaces

$$\Omega B|\Lambda|\colon ku \to |\mathcal{L}| \simeq \mathbb{Z} \times K(\mathbb{Z}, 2)$$

is the projection to the second Postnikov section in the category of ∞-loop spaces. ♦

Proof: It is an ∞-loop map by construction, so all we need to check is that it is a π_n-equivalence for $n \leq 2$.

The functor Λ sends objects $\mathbb{N}_0$ to $\mathbb{Z}$ by the standard inclusion. So we need only check that the connectivity of the map is at least two on the components corresponding to $n \in \mathbb{N}_0$ for large enough n. This corresponds to being at least 1-connective on the space of automorphisms for large n and the determinant

$$\det\colon \mathrm{Gl}_n(\mathbb{C}) \to \mathbb{C}^*$$

satisfies this. □

Definition 2.7 For any bipermutative category $\mathcal{B}$ define $\mathrm{M}_n(\mathcal{B})$ as the category with

- objects n by n matrices $E = (E_{ij})_{i,j=1}^n$ of objects in $\mathcal{B}$, and
- morphisms n by n matrices $\phi = (\phi_{ij})_{i,j=1}^n$ of morphisms in $\mathcal{B}$, with the obvious sources and targets.

We define a monoidal product on $\mathrm{M}_n(\mathcal{B})$ by

$$\cdot\colon\ \mathrm{M}_n(\mathcal{B}) \times \mathrm{M}_n(\mathcal{B}) \to \mathrm{M}_n(\mathcal{B})$$

by standard matrix multiplication formula:

$$(E \cdot F)_{ik} = \bigoplus_{j=1}^{n} (E_{ij} \otimes F_{jk}).$$

We need not specify parenthesis because $\oplus$ is strictly associative. This does, however, not in general produce a strictly associative product because this would imply both distributive laws in $\mathcal{B}$ holding strictly. But there are obvious coherency isomorphisms induced from the coherency isomorphisms in $\mathcal{B}$ making this a monoidal category - with a strict unit. ♦

We could also define sum of matrices and get a bimonoidal category, but this is not important in the following. This is because 2-vector bundles will be classified by what we could call the units of this bimonoidal category and so the product structure is the only relevant structure.

Construction 2.8 Let Λ_* denote the functor induced by Λ from $\mathrm{M}_n(\mathcal{V})$ to $\mathrm{M}_n(\mathcal{L})$. This is a strict monoidal functor because Λ is a bipermutative functor.

Even though $\mathcal{L}$ is not equipped with the trivial coherency, we may still use the fact that the operations are weakly strict to define the symmetric monoidal (with respect to $\oplus$) functor

$$i\colon \mathcal{L} \to \mathcal{L}$$

by $i(n,a) = (-n, a^{-1}) = (-1,1)\otimes(n,a)$. This is a very natural choice of inverse to $\oplus$, but be warned: it does not provide a coherent choice of inverse in the sense of [8] when passing to the classifying space $|\mathcal{L}|$. So we cannot conclude that the induced E_∞-structure is trivial, which we know it is not. We will, however, say that we have a weakly strict inverse i.

Let det be the functor from $\mathrm{M}_n(\mathcal{L})$ to $\mathcal{L}$ given by taking determinant with coefficients in $\mathcal{L}$. Since all commutative, associative and distributive laws in $\mathcal{L}$ are weakly strict and we have an weakly strict inverse i to $\oplus$ this is well-defined and sends the matrix product to the tensor product in $\mathcal{L}$. This is true independently of the unusual choice of coherency isomorphisms in $\mathcal{L}$ - because it would work with coherencies given by identities. However, as the following lemma will show det is not monoidal because of the non-trivial choice of coherency isomorphisms.

Define $\mathrm{Det} = \det \circ \Lambda_*$, again this preserves products because both det and Λ_* does so, and again the following lemma tells us that it is not monoidal. ♦

As mentioned above 2-vector bundles is related to "units" in $\mathrm{M}_n(\mathcal{V})$ and we thus need to define what we mean by this.

Definition 2.9 Let $\mathcal{L}^*$ be the full subcategory of invertible objects in $\mathcal{L}$ with respect to the product $\otimes$. I.e. using the identification in definition 2.3 we see

$$\mathrm{Mor}(\mathcal{L}^*) = \{\pm 1\} \times \mathbb{C}^*.$$

This is obviously a permutative category with respect to $\otimes$, and since the twist for $\otimes$ on the object pair -1 and -1 is not the identity we still retain part of the non-trivial coherency structure from $\mathcal{L}$ in $\mathcal{L}^*$.

Also define $\mathcal{L}^*_+$ to be the sub-category with the single object 1 and $\mathbb{C}^*$ as automorphisms. This is the usual way of identifying a group with a category, however, we have also given it the canonical permutative structure using that the product is Abelian, and it also comes with its inclusion of permutative categories into $(\mathcal{L}, \otimes)$ as the "positive" units. ♦

Definition 2.10 Let $\mathrm{Gl}_n(\mathcal{V})$ be the full sub-category of $\mathrm{M}_n(\mathcal{V})$ defined by the pre-image of $\mathcal{L}^*$ using the the functor Det. Also define $\mathrm{Gl}_n(\mathcal{L})$ by the pre-image of $\mathcal{L}^*$ using the functor det.

Similarly we may define $\mathrm{Sl}_n(\mathcal{V})$ and $\mathrm{Sl}_n(\mathcal{L})$ using $\mathcal{L}^*_+$ instead of $\mathcal{L}^*$. ♦

This definition implies that Λ_* maps $\mathrm{Gl}_n(\mathcal{V})$ to $\mathrm{Gl}_n(\mathcal{L})$, and similarly for the Sl_n's.

The definition of $\mathrm{Gl}_n(\mathcal{V})$ is equivalent to the definition in [3], because the image object in $\mathcal{L}$ of Det is the determinant of the dimension matrix. The restrictions of det and Det to these sub-categories will also be denoted det and Det. Objects in $\mathrm{Gl}_n(\mathcal{V})$ and $\mathrm{Gl}_n(\mathcal{L})$ are called weakly invertible matrices.

Lemma 2.11 The functors Det and det are not monoidal (even on the weakly invertible matrices) for $n > 1$. More precisely: when evaluated on the coherent associativity isomorphisms they produce a sign in $\mathbb{C}^*_n$, for some n, which in some cases is a minus sign. ♦

Remark 2.12 This is a very important fact and is what turns into the need for orientations on 2-vector bundles. It is highly related to the Grassmann invariant (see [5]). We plan to describe this relation better in [6]. ♦

Proof: The first statement follows from the second because in $\mathcal{L}$ we have strict (not just weakly strict) associativity, and so the appearance of a minus sign will imply that the functor does not preserve the coherency isomorphisms.

Since Λ_* is injective on objects and strict monoidal we only need to find a coherency isomorphism in $\mathrm{Gl}_n(\mathcal{L})$ involving objects in the image of Λ_*, which is sent to minus by det.

The fact that det only produces signs in $\mathbb{C}^*$ follows because any coherency isomorphism in $\mathrm{Gl}_n(\mathcal{L})$ is in each entry a coherency isomorphism from $\mathcal{L}$ which is on the form (n, s) with $n \in \mathbb{Z}$ and $s \in \{\pm 1\}$, and taking determinant involves $\oplus$ and $\otimes$ which only multiplies, divides and takes powers of the last coefficients.

An easy example of this producing a minus sign for $n = 2$ is

$$\underline{\mathrm{c}}\colon \left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\right) \cdot \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \cdot \left(\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right),$$

where $k = \mathbb{C}_k$ in $\mathcal{L}$. This is the automorphism of the object

$$\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}$$

given by the identity in all but the first entry, where it is given by the twist for $1 \oplus 1$ on the first two factors plus the identity on the last, which is -1.

$$\det(\underline{\mathrm{c}}) = (1 \cdot 3 - 2 \cdot 2, \frac{1^3(-1)^1}{1^2 1^2}) = (-1, -1) \in \mathbb{Z} \times \mathbb{C}^* = \mathrm{Mor}(\mathcal{L}).$$

This example also works for higher n simply by applying block sum with identity matrices. If one would like all determinants to take the value 1 on the objects (i.e. on the first factor above) we can take block sum with identity on the two outer matrices and take block sum with

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

on the middle factor to produce such an example. □

We will use this sign to create orientation on 2-vector bundles.

Construction 2.13 The oriented versions of $\mathrm{Gl}_n(\mathcal{L})$ ($\mathrm{Gl}_n(\mathcal{V})$ respectively) are denoted $\mathrm{OGl}_n(\mathcal{L})$ ($\mathrm{OGl}_n(\mathcal{V})$), and defined to have the same objects as the unoriented version but morphisms (f, s) where f is a morphism in the unoriented version of the category and $s \in \{\pm 1\}$.

Composition and monoidal product is given by composition and monoidal product in the original category on the first factor and in both cases multiplication on the second factor.

This would describe a trivial product of the monoidal categories $\mathrm{M}_n(\mathcal{L})$ and the monoidal category with one object and $\mathbb{Z}/2\mathbb{Z}$ as automorphisms if we did not "lift" the coherency isomorphism (or associator) in the following non-trivial way.

First define the sign $\mathrm{sgn}(\underline{\mathrm{c}})$ of an associator $\underline{\mathrm{c}}$ in $\mathrm{Gl}_n(\mathcal{L})$ ($\mathrm{Gl}_n(\mathcal{V})$) to be the unique sign such that $\det(\underline{\mathrm{c}}) = (\pm 1, \mathrm{sgn}(\underline{\mathrm{c}}))$. Then the associator in the oriented category is defined to be $\underline{\mathrm{c}}' = (\underline{\mathrm{c}}, \mathrm{sgn}(\underline{\mathrm{c}}))$, where $\underline{\mathrm{c}}$ is the associator for the same objects in the unoriented category.

These fit into the appropriate commutative diagram, i.e. the pentagon relation:

$$\begin{tikzcd}
((AB)C)D \arrow[r, "\underline{c}'_{A,B,C}\cdot \mathrm{Id}_D"] \arrow[d, "\underline{c}'_{AB,C,D}"] & (A(BC))D \arrow[r, "\underline{c}'_{A,BC,D}"] & A((BC)D) \arrow[d, "\mathrm{Id}_A \cdot \underline{c}'_{B,C,D}"] \\
(AB)(CD) \arrow[rr, "\underline{c}'_{A,B,CD}"] & & A(B(CD)).
\end{tikzcd}$$

Indeed, this is so because firstly: in the first factor we used the coherency from $\mathrm{Gl}_n(\mathcal{L})$ $(\mathrm{Gl}_n(\mathcal{V}))$ so on this factor the diagram commutes, and secondly: $\det \mathrm{Id}_D = (\pm 1, 1)$ since D is weakly invertible and tensoring with this in $\mathcal{L}$ preserves any sign in the other factor so

$$\mathrm{sgn}(\underline{c}_{A,B,C} \cdot \mathrm{Id}_D) = \mathrm{sgn}(\underline{c}_{A,B,C}), \tag{3}$$

and since det is a functor we see that the sign around the pentagon must multiply to 1. I.e. they compose to the same going from top left to bottom right. The argument is identical for Det replacing det.

So these are monoidal categories, but moreover we may define functors

$$\begin{aligned}
\mathrm{Odet}\colon \mathrm{OGl}_n(\mathcal{L}) &\to \mathcal{L}^*,\\
\mathrm{O}\Lambda_*\colon \mathrm{OGl}_n(\mathcal{V}) &\to \mathrm{OGl}_n(\mathcal{L}),\\
\mathrm{ODet}\colon \mathrm{OGl}_n(\mathcal{V}) &\to \mathcal{L}^*
\end{aligned}$$

defined on morphism by

$$\begin{aligned}
\mathrm{Odet}(f,s) &= \det(f) \otimes (1,s)\\
\mathrm{O}\Lambda_*(g,s) &= (\Lambda_*(g), s)\\
\mathrm{ODet}(g,s) &= \mathrm{Det}(g) \otimes (1,s).
\end{aligned}$$

The result of tensoring with $(1,s)$ is just multiplication with the sign s on the morphism in $\mathcal{L}^*$, which uses that we are in $\mathcal{L}^*$ and not $\mathcal{L}$. So in fact it is very important that we have restricted to the weakly invertible matrices. These oriented versions preserve products since the unoriented did and the tensor $\otimes$ in $\mathcal{L}$ is weakly strict commutative. The new and useful property is that they are in fact strict monoidal because the newly defined associators are send to identities (not symmetric monoidal since we only fix the associators). Indeed, the signs of the new associators are chosen such that they cancel with the sign that made det and Det not be monoidal.

There are also canonical strict monoidal functors:

$$\begin{aligned}
P_{\mathrm{Gl}_n(\mathcal{V})}\colon \mathrm{OGl}_n(\mathcal{V}) &\to \mathrm{Gl}_n(\mathcal{V})\\
P_{\mathrm{Gl}_n(\mathcal{L})}\colon \mathrm{OGl}_n(\mathcal{L}) &\to \mathrm{Gl}_n(\mathcal{L})
\end{aligned}$$

defined by forgetting the sign. ♦

Remark 2.14 The composite of the P's with det and Det is not the oriented functors because if we forget the sign we cannot multiply by it. However, the

diagram

$$\begin{array}{ccc} \mathrm{OGl}_n(\mathcal{V}) & \xrightarrow{\mathrm{O}\Lambda_*} & \mathrm{OGl}_n(\mathcal{L}) \\ \Big\downarrow{\scriptstyle P_{\mathrm{Gl}_n(\mathcal{V})}} & & \Big\downarrow{\scriptstyle P_{\mathrm{Gl}_n(\mathcal{L})}} \\ \mathrm{Gl}_n(\mathcal{V}) & \xrightarrow{\Lambda_*} & \mathrm{Gl}_n(\mathcal{L}) \end{array}$$

is obviously a commutative diagram of monoidal functors. ♦

All of the above categories are smooth in the sense that objects are discrete, the spaces of morphisms are smooth manifolds, and the sums and products are on morphisms spaces smooth maps. We are also in the advantages situation that all the products we work with are strictly associative and symmetrical on the level of objects - meaning that the coherency isomorphisms are automorphisms. This is so simply because we have only one object in each isomorphism class. In [2] the classifying space is defined for any 2-category. The following is a smooth version of this condensed to our case and rewritten in the language of monoidal categories. We use the smooth case only because we later wish to put smooth structures on 2-vector bundles. In the following we follow the notation for ordered open coverings in [3].

Definition 2.15 Let M be a smooth para-compact manifold with smooth partitions of unity, let $(\mathcal{B}, \cdot)$ be any smooth monoidal category with discrete objects, and let $(\mathcal{U}, \mathcal{J})$ be an ordered open cover. A smooth principle $\mathcal{B}$-bundle is

1) for each $\alpha < \beta$ in $\mathcal{J}$ an object $E^{\alpha\beta}$ in $\mathcal{B}$, such that for each $\alpha < \beta < \gamma$ we have

$$E^{\alpha\beta} \cdot E^{\beta\gamma} = E^{\alpha\gamma}$$

on the level of objects, and

2) for each $\alpha < \beta < \gamma$ we have smooth maps

$$\phi^{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \to \mathrm{Mor}(E^{\alpha\beta} \cdot E^{\beta\gamma}, E^{\alpha\gamma}) \qquad (= \mathrm{Aut}(E^{\alpha\gamma})),$$

called the **coherency maps** such that

3) the diagram

$$\begin{array}{ccccc} E^{\alpha\beta} \cdot \big(E^{\beta\gamma} \cdot E^{\gamma\delta}\big) & & \xrightarrow{\underline{c}^{\alpha\beta\gamma\delta}} & & \big(E^{\alpha\beta} \cdot E^{\beta\gamma}\big) \cdot E^{\gamma\delta} \\ {\scriptstyle \mathrm{Id}\cdot\phi^{\beta\gamma\delta}}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \phi^{\alpha\beta\gamma}\cdot\mathrm{Id}} \\ E^{\alpha\beta} \cdot E^{\beta\delta} & \xrightarrow[\phi^{\alpha\beta\delta}]{} & E^{\alpha\delta} & \xleftarrow[\phi^{\alpha\gamma\delta}]{} & E^{\alpha\gamma} \cdot E^{\gamma\delta} \end{array} \tag{4}$$

commutes for all points in each quadruple intersection $U_{\alpha\beta\gamma\delta}$.

Here $\underline{c}^{\alpha\beta\gamma\delta}$ denotes the associator for the product $\cdot$ in $\mathcal{B}$ related to the two different choices of parenthesis. The diagram may be thought of as a cocycle condition. ♦

Definition 2.16 Let $n \in \mathbb{N}_0$ be a non-negative integer. A **smooth charted (oriented) 2-vector bundle** $\mathcal{E}$ of rank n over M is a principal $\mathrm{Gl}_n(\mathcal{V})$-bundle ($\mathrm{OGl}_n(\mathcal{V})$-bundle). ♦

This is slightly different than the definition in [3], but in the unoriented case if we ignore the smoothness condition then up to the equivalence defined below (taken from [2]) this provides the same equivalence classes.

Definition 2.17 Two smooth charted (oriented) 2-vector bundles $\mathcal{E}_i, i = 0, 1$ over X are **equivalent** if they are cobordant. I.e. there exist a smooth charted (oriented) 2-vector bundle $\mathcal{E}$ over $X \times [0, 1]$ such that $\mathcal{E}_{|X\times\{t\}} = \mathcal{E}_t$ for $t = 0, 1$. Here restriction uses restriction of the ordered open cover, which removes the sets, and their indices, with empty intersection with $X \times \{i\}$, and the smooth coherency maps are assumed to be constant in the t direction close to the boundary of I - so as to make composition of bordisms well-defined in the smooth category. ♦

We use this definition, as opposed to the one in [3], even though it is less explicit, because it is easier to work with.

Definition 2.18 We say that a smooth charted 2-vector bundle is **orientable** if it is equivalent to a smooth charted 2-vector bundle which is the image of a smooth charted oriented 2-vector bundle under the strict monoidal functor $P_{\mathrm{Gl}_n(\mathcal{V})}$. ♦

Note that the definition of $\mathcal{B}$-bundle is obviously functorial with respect to strict monoidal functors, because these preserve products, associators, and compositions. This even works in the smooth setting because the functor is in fact also smooth.

Lemma 2.19 For a smooth charted 2-vector bundle the sign

$$\mathrm{sgn}(\underline{c}^{\alpha\beta\gamma\delta})$$

of the determinants of the associators defines a 3-cocycle in the Čech complex $\check{C}^*(\mathcal{U}, \{\pm 1\})$. The represented class in Čech cohomology depends only on the equivalence class of the 2-vector bundle.

Furthermore, this class is zero if and only if the vector bundle is orientable. ♦

Proof: This is virtually the same argument as in construction 2.13. Again we know from the Lane-Stasheff pentagon axiom that

$$\underline{c}^{\alpha\gamma\delta\varepsilon} \circ \underline{c}^{\alpha\beta\gamma\varepsilon} = \left(\underline{c}^{\alpha\beta\gamma\delta} \cdot \mathrm{Id}_{E^{\delta\varepsilon}}\right) \circ \underline{c}^{\alpha\beta\delta\varepsilon} \circ \left(\mathrm{Id}_{E^{\alpha\beta}} \cdot \underline{c}^{\beta\gamma\delta\varepsilon}\right).$$

Taking Det, and again using its properties (see equation 3), we get

$$\mathrm{sgn}\left(\underline{c}^{\alpha\gamma\delta\varepsilon}\right) \mathrm{sgn}\left(\underline{c}^{\alpha\beta\gamma\varepsilon}\right) = \mathrm{sgn}\left(\underline{c}^{\alpha\beta\gamma\delta}\right) \mathrm{sgn}\left(\underline{c}^{\alpha\beta\delta\varepsilon}\right) \mathrm{sgn}\left(\underline{c}^{\beta\gamma\delta\varepsilon}\right),$$

which is the co-cycle condition. Obviously the associated homology class only depends on the equivalence class since the inclusions of $X \times \{i\}$ into $X \times [0, 1]$ is a homotopy equivalence.

This class is zero if and only if there is a refinement $(\mathcal{U}, \mathcal{J})$ of the ordered open cover such that we have a chain α in $\check{C}^2(\mathcal{U}', \{\pm 1\})$ s.t. $\partial\alpha = \mathrm{Det}(\underline{c})$, but

such a choice exactly corresponds to a lift of the smooth coherency maps $\phi^{\alpha\beta\gamma}$ to $(\phi^{\alpha\beta\gamma}, \alpha)$ in the oriented category, such that they satisfy diagram 4 also in the oriented category. $\square$

Remark 2.20 This last part also tells us how many different choices of orientations there are on an orientable 2-vector bundle. ♦

Definition 2.21 A charted gerbe is a smooth charted $\mathcal{L}^*_+$-bundle. ♦

Note that since $\mathcal{L}^*_+$ has one object with automorphisms $\mathbb{C}^*$ and is strictly associative this is the same as having a standard 2-cocycle with coefficients in $\mathbb{C}^*$, and thus these are classified up to equivalence by the third cohomology class of the base manifold.

Construction 2.22 We wish to construct an inclusion of gerbes into oriented 2-vector bundles, which is a lift of the usual inclusion of gerbes into 2-vector bundles. We start by describing the usual inclusion on the level of categories.

A charted gerbe is the same as a charted $\mathrm{Sl}_1(\mathcal{L})$-bundle and it is also the same as a charted $\mathrm{Sl}_1(\mathcal{V})$-bundle. The latter seen as just a monoidal category has a natural inclusion into $\mathrm{Gl}_1(\mathcal{V})$, which in turn has a natural inclusion into $\mathrm{Gl}_n(\mathcal{V})$ by block sum with the $(n-1)\times(n-1)$ identity matrix on objects, and block sum with the identity morphism on this matrix on the morphisms. Here we call the matrix, which is the unit in the monoidal structure, the identity matrix, and of course this has an identity morphism. The explicit description of this object is a matrix with the object $\mathbb{C} \in \mathcal{V}$ on the diagonal and the object $\mathbb{C}^0 \in \mathcal{V}$ every where else. The latter object has only the identity morphism but $\mathbb{C}$ of course has other morphisms, and the identity on the matrix is just the identity in each entry.

The construction of block sum is easily seen to be a strict monoidal functor

$$\mathrm{Gl}_n(\mathcal{V}) \times \mathrm{Gl}_m(\mathcal{V}) \to \mathrm{Gl}_{n+m}(\mathcal{V}).$$

In fact it is a strict bimonoidal functor on the categories of all matrices - not just the weakly invertible, but we wont use this.

This inclusion of gerbes may be lifted to the oriented category $\mathrm{OGl}_n(\mathcal{V})$. by using the strict monoidal functor

$$: \mathrm{Gl}_1(\mathcal{V}) \to \mathrm{OGl}_1(\mathcal{V}),$$

which simply puts the sign 1 on all morphisms. This is indeed strict monoidal since the associator in $\mathrm{Gl}_1(\mathcal{V})$ is the associator for $\otimes$ in $\mathcal{V}$ on the object $\mathbb{C}$, hence it is the identity and has sign 1.

To generalize the block sum we need to incorporate the sign. So we define the functor

$$S\colon \mathrm{OGl}_n(\mathcal{V}) \times \mathrm{OGl}_m(\mathcal{V}) \to \mathrm{OGl}_{n+m}(\mathcal{V})$$

by block sum on objects, block sum on the first factor of the morphisms, and by multiplying the signs on the second factor. This is obviously a product preserving functor (by ignoring the coherencies and using weak strictness on the second factor), and for it to be strict monoidal it has to preserve the associators. In the first factor of the morphisms this follows because the above

unoriented block sum is strict monoidal, so we only need to check the sign in the second factor. This depends on the fact that we restricted to the weakly invertible matrices meaning that

$$\begin{aligned}\mathrm{Det}(S(\underline{c}_n, \underline{c}_m)) &= \mathrm{Det}(\underline{c}_n) \otimes \mathrm{Det}(\underline{c}_m) = (\pm 1, \mathrm{sgn}(\underline{c}_n)) \otimes (\pm, \mathrm{sgn}(\underline{c}_m)) = \\ &= (\pm 1, \mathrm{sgn}(\underline{c}_n)\,\mathrm{sgn}(\underline{c}_m)).\end{aligned}$$

So the sign of the block sum of two associators is the product of the signs of the associators, which is precisely what we need. The resulting inclusion functor from $\mathcal{L}^*_+$ to $\mathrm{OGl}_n(\mathcal{V})$ will be denoted i. The unoriented inclusion described above is thus the composition of functors $P_{\mathrm{Gl}_n(\mathcal{V})} \circ i$. ♦

Construction 2.23 In [3] and [2] it is described how to construct a simplicial classifying category $B\mathcal{B}$ of a monoidal category $\mathcal{B}$ such that the geometric realizations of the nerves of the categories

$$|B\mathcal{B}|$$

is a classifying space of $\mathcal{B}$-bundles. There are certain conditions that $\mathcal{B}$ must satisfy, but the categories we work with here satisfy all these. It is used repeatedly in the following that this construction is functorial from the category of monoidal categories and strict monoidal functors.

Since $\mathcal{L}^*_+$ has one object with automorphisms $\mathbb{C}^*$ it follows that $|B\mathcal{L}^*_+|$ is a $K(\mathbb{Z},3)$ and in [1] it is proven that the map

$$|B(P_{\mathrm{Gl}_n(\mathcal{V})} \circ i)|_* \colon \pi_3(|B\mathcal{L}^*_+|) \to \pi_3(|B\mathrm{Gl}_n(\mathcal{V})|)$$

sends the canonical generator to an element divisible by 2 modulo torsion. They use this as an obstruction to creating a retraction back to $|B\mathcal{L}^*_+| \simeq K(\mathbb{Z},3)$. Or more specifically a determinant like map to $|B\mathrm{Gl}_1(ku)| = B(\{\text{-}1,1\} \times BU)$, which composed with the canonical map to $K(\mathbb{Z},3)$ would yield such a retraction.

The point of the orientations is that the the monoidal functor ODet provides a retraction:

$$|B\,\mathrm{ODet}| \colon |B\mathrm{OGl}_n(\mathcal{V})| \to |B\mathcal{L}^*| \simeq K(\mathbb{Z}/2\mathbb{Z},1) \times |B\mathcal{L}^*_+| \to |B\mathcal{L}^*_+|$$

of $|Bi|$. Here the latter map is the projection and the identification

$$|B\mathcal{L}^*| \simeq K(\mathbb{Z}/2\mathbb{Z},1) \times |B\mathcal{L}^*_+|$$

is due to the fact that the monoidal structure in $\mathcal{L}^*$ is strictly associative, and so the A_∞ structure splits. This splitting can be described using monoidal functors in the following way: we have the inclusion

$$\mathcal{L}^*_+ \to \mathcal{L}^*,$$

which is a strict symmetric monoidal functor, and we have a left inverse (or projection)

$$p \colon \mathcal{L}^* \to \mathcal{L}^*_+ \tag{5}$$

given by $p(d,a) = (1, a^d)$ this is not symmetric monoidal because the symmetry on -1 is not the identity. It is, however, strict monoidal because

$$p((d,a) \otimes (e,b)) = p(de, a^e b^d) = (1, a^d b^e) = \\ = (1, a^d) \otimes (1, b^e) = p((d,a)) \otimes p((e,b)),$$

and all associators are identities.

We now see that $p \circ \mathrm{ODet}$ is a strict monoidal functor and right inverse to the strict monoidal functor i.

Definition 2.24 We define the strict monoidal functor Dger from $\mathrm{OGl}_n(\mathcal{V})$ to $\mathcal{L}_+^*$ as the composite $p \circ \mathrm{ODet}$. ♦

We thus conclude that we have removed the obstruction and that the half magnetic monopole in [1] must be an unorientable 2-vector bundle.

3 Connective structures

In [4] Brylinski defines gerbes over a space X and classify their equivalence classes by $H^3(X, \mathbb{Z})$. He also gives an example in chapter 7 of how to get from a charted gerbe to his definition of a gerbe, which is easily generalized. He defines the notion of a connective structure on a gerbe, and in the charted case this corresponds to having Hermitian connections $\nabla_{\alpha\beta}$ on the trivial line bundles

$$L_{\alpha\beta} = U_{\alpha\beta} \times \mathbb{C}$$

such that the coherency maps $\phi^{\alpha\beta\gamma} \colon \mathbb{C}^* \to \mathbb{C}^*$ describes isometries of line bundles

$$L_{\alpha\beta|U_{\alpha\beta\gamma}} \otimes L_{\beta\gamma|U_{\alpha\beta\gamma}} \to L_{\alpha\gamma|U_{\alpha\beta\gamma}}.$$

We now generalize this notion of connective structures to 2-vector bundles.

Definition 3.1 A connective structure ∇ on a smooth charted two vector bundle $\mathcal{E}$ is for each $\alpha < \beta$ in $\mathcal{J}$ a choice of an $n \times n$-matrix of connections $\nabla^{\alpha\beta}$ on the matrix of trivial bundles

$$U_{\alpha\beta} \times E^{\alpha\beta},$$

such that for all $\alpha < \beta < \gamma$ the pullback $(\phi^{\alpha\beta\gamma})^* \nabla^{\alpha\gamma}$ is the same connection as the one induced from the product $E^{\alpha\beta} \cdot E^{\beta\gamma}$ of matrices in $\mathrm{M}_n(\mathcal{V})$. This we write as

$$(\phi^{\alpha\beta\gamma})^* \nabla^{\alpha\gamma} = \nabla^{\alpha\beta} \cdot \nabla^{\beta\gamma},$$

and diagram 4 tells us that

$$(\phi^{\alpha\gamma\delta-1})^* [(\phi^{\alpha\beta\gamma-1})^* (\nabla^{\alpha\beta} \cdot \nabla^{\beta\gamma}) \cdot \nabla^{\gamma\delta}] = \\ (\phi^{\alpha\beta\delta-1})^* [\nabla^{\alpha\beta} \cdot (\phi^{\beta\gamma\delta-1})^* (\nabla^{\beta\gamma} \cdot \nabla^{\gamma\delta})].$$

In light of this we define for any such connections the convenient associative "product"

$$\nabla^{\alpha\beta}\bullet\nabla^{\beta\gamma} = ((\phi^{\alpha\beta\gamma})^{-1})^*(\nabla^{\alpha\beta}\cdot\nabla^{\beta\gamma}),$$

which is a connection on $U_{\alpha\beta\gamma}\times E^{\alpha\gamma}$. In this notation the requirement for a family of connections to be a connective structure is the cocycle condition with respect to $\bullet$. ♦

This product behaves well under smooth convex combinations: that is

$$\begin{aligned}&(\psi_1\nabla_1^{\alpha\beta}+\psi_2\nabla_2^{\alpha\beta})\bullet(\psi_1'\nabla_1^{\beta\gamma}+\psi_2'\nabla_2^{\beta\gamma}) = \\ =&\psi_1\psi_1'\nabla_1^{\alpha\beta}\bullet\nabla_1^{\beta\gamma}+\psi_2\psi_1'\nabla_2^{\alpha\beta}\bullet\nabla_1^{\beta\gamma}+\psi_1\psi_2'\nabla_1^{\alpha\beta}\bullet\nabla_2^{\beta\gamma}+\psi_2\psi_2'\nabla_2^{\alpha\beta}\bullet\nabla_2^{\beta\gamma},\end{aligned}\quad (6)$$

and

$$(\psi_1\nabla_1^{\alpha\beta}+\psi_2\nabla_2^{\alpha\beta})\bullet(\psi_1\nabla_1^{\beta\gamma}+\psi_2\nabla_2^{\beta\gamma}) = \psi_1\nabla_1^{\alpha\beta}\bullet\nabla_1^{\beta\gamma}+\psi_2\nabla_2^{\alpha\beta}\bullet\nabla_2^{\beta\gamma}\quad (7)$$

are true for any smooth functions $\psi_1+\psi_2=1$ and $\psi_1'+\psi_2'=1$. Indeed, they are true for the tensor product of vector bundles, and this we may use on each direct summand of each entry in the matrix, and pullback preserves convex combinations.

Lemma 3.2 Let $\mathcal{E}$ be any smooth charted 2-vector bundle. After a possible elementary refinement $\mathcal{E}$ has a connective structure and such a choice is a contractible choice. ♦

Proof: For any partially ordered set $\mathcal{J}$ define

$$\mathcal{J}_\alpha^\beta = \{\alpha_0<\cdots<\alpha_k \mid k\in\mathbb{N}, \alpha_0=\alpha, \alpha_k=\beta\}$$

the finite sequences in $\mathcal{J}$ connecting α and β. By para-compactness we may assume after refinement that

C1) the cover $(\mathcal{U},\mathcal{J})$ is locally finite.

Again using para-compactness we shrink $(\mathcal{U},\mathcal{J})$ to an ordered open cover $(\mathcal{U}',\mathcal{J})$ (with carrier function the identity), such that

C2) for any $\alpha\in\mathcal{J}$ we have the closure of U_α' contained in U_α.

Now we use existence of smooth partition of unity to get smooth functions

$$\psi_\alpha\colon M\to[0,1]$$

with $\psi_{\alpha|U_\alpha'}=1$ and $\psi_{\alpha|M-U_\alpha}=0$.

Choose any connections $\nabla_0^{\alpha\beta}$ on

$$U_{\alpha\beta}\times E^{\alpha\beta}.$$

For each $\overrightarrow{\alpha}\in\mathcal{J}_\alpha^\beta$ we have a connection

$$\nabla_0^{\overrightarrow{\alpha}} = \nabla_0^{\alpha_0\alpha_1}\bullet\cdots\bullet\nabla_0^{\alpha_i\alpha_{i+1}}\bullet\cdots\bullet\nabla_0^{\alpha_{k-1}\alpha_k}$$

defined on $U_{\alpha_0 \dots \alpha_k} \subset U_{\alpha\beta}$. We will smoothly interpolate between these. So we define the weights:

$$\psi_{\overrightarrow{\alpha}} = \prod_{i=0}^{k-1} \left(\psi_{\alpha_i} \psi_{\alpha_{i+1}} \prod_{\alpha_i < \gamma < \alpha_{i+1}} (1 - \psi_\gamma) \right),$$

for each $\overrightarrow{\alpha} \in \mathcal{J}_\alpha^\beta$. The product is well-defined and smooth because C1) implies locally finiteness of the second product. Notice that

$$\psi_{\overrightarrow{\alpha}} = \psi_{\overrightarrow{\alpha}_1} \psi_{\overrightarrow{\alpha}_2}, \tag{8}$$

when $\alpha < \beta < \gamma$, $\overrightarrow{\alpha}_1 \in \mathcal{J}_\alpha^\beta$, $\overrightarrow{\alpha}_2 \in \mathcal{J}_\beta^\gamma$, and $\overrightarrow{\alpha} \in \mathcal{J}_\alpha^\gamma$ is the obvious concatenation of $\overrightarrow{\alpha}_1$ and $\overrightarrow{\alpha}_2$.

On the sets $U'_{\alpha\beta} \subset U_{\alpha\beta}$ we then define

$$\nabla^{\alpha\beta} = \frac{\sum_{\overrightarrow{\alpha} \in \mathcal{J}_\alpha^\beta} (\psi_{\overrightarrow{\alpha}} \nabla_0^{\overrightarrow{\alpha}})}{\sum_{\overrightarrow{\alpha} \in \mathcal{J}_\alpha^\beta} (\psi_{\overrightarrow{\alpha}})},$$

This is well-defined and smooth because;

- the weights in the sum are 0 when the connections are not defined,
- C1) implies that the sums are locally finite,
- and $\psi_{\overrightarrow{\alpha}}$ is non-zero when we include the γ's with $\psi_\gamma = 1$ and exclude those with $\psi_\gamma = 0$ in the sequence $\overrightarrow{\alpha}$ - notice in particular that $\psi_\alpha = \psi_\beta = 1$ because we restricted to the set $U'_{\alpha\beta}$.

It satisfies the wanted cocycle condition because: at any point $x \in U'_{\alpha\beta\gamma}$ for $\alpha < \beta < \gamma$ in $\mathcal{J}$ we have $\psi_\alpha = \psi_\beta = \psi_\gamma = 1$ and so β has to be included in the sequence $\overrightarrow{\alpha} = (\alpha = \alpha_0 < \cdots < \alpha_k = \gamma)$ for $\psi_{\overrightarrow{\alpha}}$ to be non-zero. Indeed, we use that the subset $\mathcal{J}_x \subset \mathcal{J}$ defined by $\mathcal{J}_x = \{\delta \in \mathcal{J} \mid x \in U_\delta\}$ is totally ordered to conclude that if β is not in $\overrightarrow{\alpha}$ then the factor $(1 - \psi_\beta)$ is part of the product defining the weight $\psi_{\overrightarrow{\alpha}}$, which is thus 0. Now we may use (6) and (8) repeatedly to see

$$\nabla^{\alpha\gamma} = \frac{\sum_{\mathcal{J}_\alpha^\gamma} \psi_{\overrightarrow{\alpha}} \nabla_0^{\overrightarrow{\alpha}}}{\sum_{\mathcal{J}_\alpha^\gamma} \psi_{\overrightarrow{\alpha}}} = \frac{\left(\sum_{\mathcal{J}_\alpha^\beta} \psi_{\overrightarrow{\alpha}} \nabla_0^{\overrightarrow{\alpha}} \right) \bullet \left(\sum_{\mathcal{J}_\beta^\gamma} \psi_{\overrightarrow{\alpha}} \nabla_0^{\overrightarrow{\alpha}} \right)}{\left(\sum_{\mathcal{J}_\alpha^\beta} \psi_{\overrightarrow{\alpha}} \right) \left(\sum_{\mathcal{J}_\beta^\gamma} \psi_{\overrightarrow{\alpha}} \right)} = \nabla^{\alpha\beta} \bullet \nabla^{\beta\gamma}.$$

The choice is contractible because given two connective structures ∇ and ∇' equation (7) tells us that $t\nabla + (1-t)\nabla'$ is a connective structure (this convex combination should be interpreted over every double intersection). $\square$

As constructed in the end of the previous section we can use the functor Dger to map charted oriented 2-vector bundles to charted gerbes. We will enrich this map such that it carries connective structures to connective structures.

Construction 3.3 Let $\mathcal{E}$ be a charted oriented 2-vector bundle. Taking the functor Dger from definition 2.24 on this charted bundle produces a 2-cocycle with coefficients in $\mathbb{C}^*$. Indeed, this is because the category $\mathcal{L}_+^*$ to which Dger maps is the strict monoidal category with one object, $\mathbb{C}^*$ as its automorphisms and monoidal product the same as composition.

To get a connective structure we need a connection on each line bundle compatible with the isomorphisms. We construct this also using the functor Dger: we simply define the parallel transport in $U_{\alpha\beta}\times\mathbb{C}$ along a path $f\colon [0,1]\to U_{\alpha\beta}$ by taking the functor Dger of the automorphism $P_f^{\alpha\beta}$ defined on $E^{\alpha\beta}$ by parallel transport in

$$U_{\alpha\beta}\times E^{\alpha\beta}$$

along the path f using the connections $\nabla^{\alpha\beta}$. Since the connection matrices are preserved using the isomorphisms $\phi^{\alpha\beta\gamma}$ we conclude that the parallel transport morphisms are preserved. So

$$\begin{array}{ccc} E^{\alpha\beta}\cdot E^{\beta\gamma} & \xrightarrow{\phi^{\alpha\beta\gamma}(f(0))} & E^{\alpha\gamma} \\ {\scriptstyle P_f^{\alpha\beta}\cdot P_f^{\beta\gamma}}\downarrow & & \downarrow{\scriptstyle P_f^{\alpha\gamma}} \\ E^{\alpha\beta}\cdot E^{\beta\gamma} & \xrightarrow{\phi^{\alpha\beta\gamma}(f(1))} & E^{\alpha\gamma} \end{array}$$

Commutes. This implies that

$$\begin{array}{ccc} \mathbb{C}_1 & \xrightarrow{\mathrm{Dger}(\phi^{\alpha\beta\gamma}(f(0)))} & \mathbb{C}_1 \\ {\scriptstyle \mathrm{Dger}(P_f^{\alpha\beta})\otimes\mathrm{Dger}(P_f^{\beta\gamma})}\downarrow & & \downarrow{\scriptstyle \mathrm{Dger}(P_f^{\alpha\gamma})} \\ \mathbb{C}_1 & \xrightarrow{\mathrm{Dger}(\phi^{\alpha\beta\gamma}(f(1)))} & \mathbb{C}_1 \end{array}$$

commutes because Dger is strict monoidal. The monoidal product $\otimes$ on $\mathcal{L}_+^*$ are multiplication in $\mathbb{C}^*$. So the induced connections on the line bundles are compatible with the induced 2-cocycle. ♦

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